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Projections onto Convex Cones in Hilbert Space

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Given a closed convex cone C in a Hilbert space H, we investigate the function which assigns to each point x in H the nearest point of C to x. We call this function the *projection* of H onto C and we give an algebraic characterization of this function which generalizes the well-known characterization of a projection onto a closed subspace as an idempotent, symmetric linear operator.

I. INTRODUCTION

Let H be a real Hilbert space. The subset C of H is a *Chebyshev set* if for each point x of H, there is a unique point of C which is nearest to x; i.e., there is a point z in C such that $\|x-z\| < \|x-y\|$ for every $y \in C\setminus\{z\}$. The point z is sometimes called the point of best approximation in C to x. The set C is said to be *convex* if $tx+(1-t)y \in C$ whenever $x, y \in C$ and $0 \leqslant t \leqslant 1$. It is well-known that every closed, convex set in a Hilbert space is a Chebyshev set. It remains unknown if the converse (see Question 1 below) is true.

QUESTION 1. *If C is a Chebyshev set in a Hilbert space, is C closed and convex?*

It is immediate that C is closed; so convexity is the real issue. A number of people have given positive answers to Question 1 in a finite dimensional setting. See, for example, [4, 9, 10, 12, 15, 19]. V. L. Klee [13] has the most general positive result in an infinite dimensional setting. He shows that weakly closed and Chebyshev implies convexity. G. G. Johnson [11] has an example of a non-convex Chebyshev set in a non-complete inner product space. It is not known if his example can be extended to the completion of the space. Some other results related to this question can be

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found in [1, 3, 16, 17]. Thorough discussions of this question can be found in [6; 19, Sect. 2; 20].

If C is a Chebyshev set, there is a natural surjective function $p: H \to C$ such that, for each $x \in H$, $p(x)$ is the unique nearest point of C to x. We will call p the *projection* of H onto C. Other authors have called p a proximity or near-point function. The focus of this paper is to study the algebraic characteristics of projections onto certain Chebyshev sets in Hilbert spaces. In particular, we will give an algebraic characterization of projections onto closed convex cones. This result generalizes the well-known characterization of projections onto closed subspaces as idempotent, symmetric, linear operators (see [14, p. 394] or [18, p. 299]). While this characterization is of interest in its own right, a similar characterization, if one exists, of projections onto closed convex sets might provide a new approach to answering Question 1.

Some other results related to projection mappings onto closed convex cones can be found in [2, 7, 22, 23, 24].

II. Convex Cones and Projections

A *Hilbert space H* is a complete inner product space. A non-empty subset of H is a *convex cone* if it is closed under addition and closed under multiplication by positive scalars. We will also assume that $\mathbf{0}$ is an element of all cones under consideration in this paper. Linear subspaces are convex cones and convex cones are clearly convex. We begin by stating some elementary results about convex cones and projections onto closed convex cones. These results will be of use to us later in the paper.

Lemma 1. *Let C be a convex cone.*

(i) *If $x \in H$, $y \in C$, and there exist $\alpha > 0$ and $\beta \leqslant 0$ such that $\alpha x + \beta y \in C$, then $x \in C$.*

(ii) *Let $x, y \in H$. There exist $\alpha, \beta > 0$ such that $\alpha x + \beta y \in C$ if and only if there exist $0 < t < 1$ such that $tx + (1-t)y \in C$.*

Proof. (i) Now, $1/\alpha > 0$ and $-\beta/\alpha \geqslant 0$, so $x = (1/\alpha)(\alpha x + \beta y) + (-\beta/\alpha)y \in C$.

(ii) Suppose there exist $\alpha > 0$ and $\beta > 0$ such that $\alpha x + \beta y \in C$. Let $t = \alpha/(\alpha + \beta)$. Then $0 < t < 1$ and $tx + (1-t)y = (\alpha/(\alpha+\beta))x + (\beta/(\alpha+\beta))y = (1/(\alpha+\beta))(\alpha x + \beta y) \in C$. The other direction is trivial.

For $C \subseteq H$, let $C^* = \{x \in H \mid \langle x, y \rangle \leqslant 0 \text{ for every } y \in C\}$. ($C^*$ is called the *dual cone* of C.) The results in the next lemma are well-known, so we omit the proofs.

LEMMA 2. *Let $C \subseteq H$.*

(i) *C^* is a closed, convex cone.*

(ii) *C is a closed, convex cone if and only if $C = C^{**}$.*

(iii) *Let C be a closed, convex cone and $x \in H$. Then there is a unique $x_1 \in C$ and a unique $x_2 \in C^*$ such that $x = x_1 + x_2$ and $\langle x_1, x_2 \rangle = 0$. ($x_1$ is the nearest point of C to x and x_2 is the nearest point of C^* to x.)*

LEMMA 3. *Let C be a closed convex cone, p the projection of H onto C, and $x, y \in H$. Then*

(i) *$p^2 = p$ (p is idempotent).*

(ii) *$p(\alpha x) = \alpha p(x)$ for $\alpha \geqslant 0$ (p is non-negatively homogeneous).*

(iii) *$p(x+y) = p(x) + p(y)$ if and only if $\langle p(x), y \rangle = \langle p(x), p(y) \rangle = \langle x, p(y) \rangle$.*

(iv) *$\langle x - p(x), p(y) \rangle \leqslant 0$.*

(v) *$\langle x - p(x), p(x) \rangle = 0$.*

(vi) *$\| p(x) - p(y) \| \leqslant \| x - y \|$ (p is non-expansive).*

(vii) *$I - p$ is the projection of H onto C^*.*

(viii) *$C^* = p^{-1}(\mathbf{0})$.*

Proof. With the possible exception of part (iii), all of these are well-known facts. Therefore, we will prove only part (iii). Proofs for parts (i) and (ii) are easy. Parts (iv) and (v) actually characterize the element $p(x)$ in C which is the projection of x onto C (see [8, Prop. 1.12.4]). That p is non-expansive can be found in [5, Th. 3]. Proofs for parts (vii) and (viii) are straightforward.

Proof of (iii). We first note that by part (v), we have that for $x, y \in H$,

$$\begin{aligned} &\langle p(y), x - p(x) \rangle + \langle p(x), y - p(y) \rangle \\ &\quad = \langle p(x), x - p(x) \rangle + \langle p(y), x - p(x) \rangle \\ &\qquad + \langle p(x), y - p(y) \rangle + \langle p(y), y - p(y) \rangle \\ &\quad = \langle p(x) + p(y), x + y - (p(x) + p(y)) \rangle. \end{aligned} \tag{$*$}$$

Now suppose $x, y \in H$ and $p(x+y) = p(x) + p(y)$. Then by ($*$) and part (v), we have

$$\langle p(y), x - p(x) \rangle + \langle p(x), y - p(y) \rangle = \langle p(x+y), x + y - p(x+y) \rangle = 0.$$

Since both $\langle p(y), x - p(x) \rangle$ and $\langle p(x), y - p(y) \rangle$ are non-positive by part (iv), this implies that $\langle p(y), x - p(x) \rangle = 0 = \langle p(x), y - p(y) \rangle$.

Now suppose $\langle p(y), x-p(x)\rangle = 0 = \langle p(x), y-p(y)\rangle$. Write

$$x+y=p(x)+p(y)+(x+y-(p(x)+p(y))),$$

where $p(x)+p(y)\in C$, and $x+y-(p(x)+p(y))\in C^*$ by part (vii) and Lemma 2, part (i). Also, $\langle p(x)+p(y), x+y-(p(x)+p(y))\rangle = 0$ by (*) and our assumption. Observe that

$$x+y=p(x+y)+(x+y-p(x+y))$$

is also a representation of $x+y$ in the form given by Lemma 2, part (iii). By the uniqueness of this type of representation, we must have that $p(x+y)=p(x)+p(y)$.

III. Idempotent, Positively Homogeneous, Face-Linear Transformations

We will focus on properties (i), (ii), and (iii) of Lemma 3. They are analogous to the idempotent, symmetric, and linear properties which characterize projections onto closed subspaces. In particular, property (iii) says that a projection onto a closed, convex cone is additive with respect to two points if and only if it is symmetric with respect to these points. We will say that a function which satisfies this property is *face-linear*. For the remainder of this paper, we let $n: H\to H$ denote an arbitrary function which satisfies the following three properties. For x, $y\in H$,

(1) $n^2(x)=n(x)$,

(2) $n(\alpha x)=\alpha n(x)$ for $\alpha>0$, and

(3) $n(x+y)=n(x)+n(y)$ if and only if $\langle n(x), y\rangle = \langle n(x), n(y)\rangle = \langle x, n(y)\rangle$.

For convenience, we let $K=n(H)$. For $M\subseteq H$, let $\bar{M}$ and $F(M)$ denote respectively the topological closure and boundary of M. We now consider the following question.

Question 2. *Is K a closed, convex cone and, if so, is n the projection of H onto K?*

We begin by looking at some of the properties that n and K must have.

Theorem 1.1. *Let x, $y\in H$. Then*

(i) $\langle x-n(x), n(x)\rangle = 0$.

(ii) $\|n(x)\|\leqslant\|x\|$. *Furthermore,* $\|n(x)\|=\|x\|$ *if and only if* $n(x)=x$ *if and only if* $x\in K$; *and* $\|n(x)\|<\|x\|$ *if and only if* $x\notin K$.

(iii) $n(\mathbf{0})=\mathbf{0}$.

(iv) *K is a convex cone.*

(v) *If $n(x+y)=n(x)+n(y)$, then, for every $\alpha, \beta \geqslant 0$, $n(\alpha x+\beta y)= n(\alpha n(x)+\beta y)=n(\alpha x+\beta n(y))=n(\alpha n(x)+\beta n(y))=\alpha n(x)+\beta n(y)$.*

(vi) *If $n(x)=n(y)$, then, for $\alpha, \beta \geqslant 0$, $n(\alpha x+\beta y)=(\alpha+\beta)\, n(x)$. In particular, $n(\alpha x+\beta n(x))=(\alpha+\beta)\, n(x)$.*

(vii) *If $x \in n^{-1}(\mathbf{0})$, $y \in K$, and $\langle x, y\rangle=0$, then, for $\alpha, \beta \geqslant 0$, $n(\alpha x+\beta y)=\beta y$.*

(viii) *If $x \in n^{-1}(\mathbf{0})$, $y \in K$, $\langle x, y\rangle=0$, and $\alpha x+\beta y \in K$ for some $\alpha>0$ and $\beta \geqslant 0$, then $x=\mathbf{0}$.*

(ix) *If $\alpha x+\beta n(x) \in K$ for some $\alpha>0$ and $\beta \in \mathbf{R}$, then $x \in K$.*

(x) *If $x \notin K$, then $n(x) \in F(K)$.*

(xi) $K^* \subseteq n^{-1}(\mathbf{0})$.

Proof. (i) By property (2), $n(x+x)=n(2x)=2n(x)=n(x)+n(x)$. By property (3), this implies that $\langle n(x), x\rangle=\langle n(x), n(x)\rangle$.

(ii) By part (i), we get that $\|x\|^2=\|x-n(x)\|^2+\|n(x)\|^2$. This implies that $\|n(x)\| \leqslant \|x\|$. If $\|n(x)\|=\|x\|$, then we get that $\|x-n(x)\|=0$; and thus $n(x)=x$. If $x \in K$, then $x=n(y)$ for some $y \in H$. By property (1), this implies that $n(x)=n(n(y))=n(y)=x$. The other implications of (ii) are trivial.

(iii) By part (ii), $\|n(\mathbf{0})\| \leqslant \|\mathbf{0}\|=0$. Thus, $n(\mathbf{0})=\mathbf{0}$.

(iv) Suppose $x, y \in K$ and $\alpha \geqslant 0$. By part (ii), $n(x)=x$ and $n(y)=y$. Hence, each of $\langle n(x), y\rangle$, $\langle n(x), n(y)\rangle$, and $\langle x, n(y)\rangle$ is equal to $\langle x, y\rangle$. We have that $n(\alpha x)=\alpha n(x)=\alpha x$ by property (2), and $n(x+y)= n(x)+n(y)$ by property (3). From part (ii) again, we have that $\alpha x, x+y \in K$. Thus, K is a convex cone.

(v) By property (3), $\langle n(x), y\rangle=\langle n(x), n(y)\rangle=\langle x, n(y)\rangle$. By property (2), we get that

$$\langle n(\alpha x), \beta y\rangle=\alpha\beta\langle n(x), y\rangle,$$
$$\langle n(\alpha x), n(\beta y)\rangle=\alpha\beta\langle n(x), n(y)\rangle,$$
$$\langle \alpha x, n(\beta y)\rangle=\alpha\beta\langle x, n(y)\rangle.$$

Thus, $\langle n(\alpha x), \beta y)\rangle=\langle n(\alpha x), n(\beta y)\rangle=\langle \alpha x, n(\beta y)\rangle$. By properties (3) and (2), this implies that $n(\alpha x+\beta y)=n(\alpha x)+n(\beta y)=\alpha n(x)+\beta n(y)$. Using properties (1) and (2), the other equalities in this part follow in a similar manner.

(vi) By part (i), each of $\langle n(x), y\rangle$, $\langle n(x), n(y)\rangle$, and $\langle x, n(y)\rangle$ is equal to $\|n(x)\|^2$. By property (3), this implies that $n(x+y) = n(x)+n(y) = 2n(x)$. This part now follows from part (v).

(vii) This statement follows from property (3) and from (v) above.

(viii) By (vii), $n(\alpha x+\beta y) = \beta y$. Since $\alpha x+\beta y \in K$, it follows that $n(\alpha x+\beta y) = \alpha x+\beta y$. Thus, $\alpha x = \mathbf{0}$; and therefore, $x = \mathbf{0}$.

(ix) If $\beta \geqslant 0$, it follows from (vi) that $n(\alpha x+\beta n(x)) = (\alpha+\beta)\, n(x)$. Since $\alpha x+\beta n(x) \in K$, we get that $n(\alpha x+\beta n(x)) = \alpha x+\beta n(x)$. Hence, $x = n(x)$, and thus $x \in K$. If $\beta < 0$, let $t = 1/(1-\beta)$. Clearly $0 < t < 1$. Since K is convex, $t(\alpha x+\beta n(x)) + (1-t)\, n(x) \in K$. Now, $t(\alpha x+\beta n(x)) + (1-t)\, n(x) = (t\alpha)x$. Hence, since K is a cone, $x \in K$.

(x) Since $x \notin K$, it follows from (ix) that for each $\alpha > 0$, $\alpha x + (1-\alpha)\, n(x) \notin K$. Hence, $n(x) \in F(K)$.

(xi) Suppose $x \in K^*$. Then by part (i), $|\, n(x)\,|^2 = \langle x, n(x)\rangle \leqslant 0$ (since $n(x) \in K$). Thus, $n(x) = \mathbf{0}$, and hence $x \in n^{-1}(\mathbf{0})$.

Since the image of n, namely K, must be a convex cone, we let p be the projection of H onto $\bar{K}$ and we note two relationships between n and p.

THEOREM 1.2. *Let $x \in H$. Then*

(i) *If $p(x) \in n^{-1}(\mathbf{0})$, then, for $\alpha, \beta \geqslant 0$, $\alpha p(x) + \beta(x - p(x)) \in n^{-1}(\mathbf{0})$.*

(ii) *If $p(x) \in K$, then $n(x) = p(x)$.*

Proof. (i) By Lemma 3, part (vii), $x - p(x) \in \bar{K}^* = K^*$. By Theorem 1.1, part (xi), $n(x - p(x)) = \mathbf{0}$. Thus, by Theorem 1.1, part (vi), $n(\alpha p(x) + \beta(x - p(x))) = (\alpha+\beta)\, n(p(x)) = \mathbf{0}$.

(ii) As in (i) above, $n(x - p(x)) = \mathbf{0}$. Also, by Lemma 3, part (v), $\langle x - p(x), p(x)\rangle = \mathbf{0}$. Thus, by Theorem 1.1, part (vii), $n(x) = n((x - p(x)) + p(x)) = p(x)$.

Note that if p is the projection of H onto a closed, convex cone, then p also has all of the properties listed in Theorem 1.1. It is not immediately apparent whether properties (1), (2), and (3) imply that K is closed or that n is the projection of H onto K. The next result shows that if K is closed, then we do have that n is the projection of H onto K.

THEOREM 2. *Suppose that $n\colon H \to H$ is a function, $C = n(H)$, and C is closed. Then C is a closed, convex cone and n is the projection of H onto C if and only if n satisfies properties* (1), (2), *and* (3).

Proof. If n satisfies properties (1), (2), and (3), then by Theorem 1.1, part (iv), C is a closed, convex cone. By Theorem 1.2, part (ii), $p(x) = n(x)$ for all $x \in H$, where p is the projection mapping of H onto $\bar{C} = C$. Hence, n is the projection mapping of H onto C.

The opposite implication is immediate from Lemma 3.

COROLLARY 1. *Suppose that* $n: H \to H$ *is a function and* $C = n(H)$. *Then* C *is a closed, convex cone and* n *is the projection mapping of* H *onto* C *if and only if* n *is a continuous function satisfying properties* (1), (2), *and* (3).

Proof. Suppose n is continuous and satisfies (1), (2), and (3). It follows from continuity and idempotentness that C is closed. Thus, by Theorem 2, C is a closed convex cone and n is the projection of H onto C.

The opposite implication again follows from Lemma 3.

We point out that if n is the projection mapping onto a closed convex cone, then it follows from Lemma 3(vi) that n is continuous. It would be of interest to know if one must assume continuity of n to obtain the opposite implication in the characterization given in Corollary 1. Thus, we ask the following question.

QUESTION 3.1. *In Corollary* 1, *can the assumption that n is continuous be omitted?*

It is equivalent to ask the following.

QUESTION 3.2. *In Theorem* 2, *can the assumption that* C *is closed be omitted?*

If properties (2) and (3) of n are replaced with "n is symmetric and linear," it is well-known that these conditions characterize n as the projection onto a closed linear subspace of H. In this setting, the assumption that n is continuous (or that $n(H)$ is closed) is not necessary.

The second author can show that the answer to Question 3.1 (or Question 3.2) is YES if $\dim(H) \leqslant 3$.

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